DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

262. Proposed by REV. R. D. CARMICHAEL, Hartselle, Ala.

Sum to infinity the series $\dfrac{n}{(4n^2-1)^2}$, beginning with $n=1$, n being always odd.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\frac{n}{(4n^2-1)^2}=\tfrac{1}{8}\left(\frac{1}{(2n-1)^2}-\frac{1}{(2n+1)^2}\right).$$

When $n=1, 3, 5, 7, \ldots$

$$\Sigma\frac{n}{(4n^2-1)^2}=\tfrac{1}{8}\left(\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}+\ldots-\frac{1}{3^2}-\frac{1}{7^2}-\frac{1}{11^2}-\ldots\right)$$

$$=\tfrac{1}{8}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right)-\tfrac{1}{4}\left(\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}+\ldots\right)$$

$$=\frac{\pi^2}{64}-\tfrac{1}{4}(.1579+\ )=\frac{\pi^2}{64}-\tfrac{1}{4}.\frac{2\pi^2}{125}\text{ nearly, }=\frac{\pi^2}{64}-\frac{\pi^2}{250}=\frac{93\pi^2}{8000}.$$

We may also write

$$\Sigma\frac{n}{(4n^2-1)^2}=\sum_{m=1}^{\infty}\frac{2m-1}{(16m^2-16m+3)^2}=\tfrac{1}{8}\int_0^1\frac{\tan^{-1}x}{x}dx.$$

This series is discussed by William E. Heal in Vol. IX, pp. 47—49, of the MONTHLY.*

Similar approximations were obtained by S. A. Corey, G. W. Greenwood, and J. Scheffer.

263. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the transcendentals e and π in the form of infinite continued fractions.

*See also an article entitled "Note on the Numerical Transcendants S_n and $s_n=S_n-1$," by Professor W. Woolsey Johnson, in the current *Bulletin of the American Mathematical Society*.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

According to a method due to Euler the series

$$S=\frac{1}{A}-\frac{1}{B}+\frac{1}{C}-\frac{1}{D}+\frac{1}{E}-\dots$$

may be converted into a continued fraction thus: Putting

$$S_1=\frac{1}{B}-\frac{1}{C}+\frac{1}{D}-\frac{1}{E}+\dots \quad S_2=\frac{1}{C}-\frac{1}{D}+\frac{1}{E}-\frac{1}{F}+\dots$$

$$S_3=\frac{1}{D}-\frac{1}{E}+\frac{1}{F}-\frac{1}{G}+\dots \text{ etc., we get}$$

$$S=\frac{1}{A}-S_1=\frac{1-A\,S_1}{A};\ \therefore\frac{1}{S}=\frac{A}{1-AS_1}=A+\frac{A^2S_1}{1-AS_1}=A+\frac{A^2}{-A+(1/S_1)},\text{etc.}$$

Thus, $S=\cfrac{1}{A+\cfrac{A^2}{B-A+\cfrac{B^2}{C-B+\cfrac{C^2}{C-D+\cfrac{D^2}{E-D+\dots}}}}}\ \dots\text{(I)}.$

Since $\frac{1}{4}\pi=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$ we get, substituting in (I), $A=1$, $B=3$, $C=5$, $D=1$, etc.,

$$\tfrac{1}{4}\pi=\cfrac{1}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\dots}}}}$$

To convert the series $\frac{1}{a}-\frac{1}{ab}+\frac{1}{abc}-\frac{1}{abcd}+\dots$ into a continued fraction, we put in (I), $A=a$, $B=ab$, $C=abc$, $D=abcd$, etc., and thus we obtain

$$\frac{1}{a}-\frac{1}{ab}+\frac{1}{abc}-\frac{1}{abcd}+\dots=\cfrac{1}{a+\cfrac{a}{b-1+\cfrac{b}{c-1+\cfrac{c}{d-1+\dots}}}}\ \dots\text{(II)}.$$

To convert $\frac{1}{a}+\frac{1}{ab}+\frac{1}{abc}+\frac{1}{abcd}+\dots$ into a continued fraction, we have in (II) only to put $-b$, $-c$, $-d$, ... for b, c, d, ... and thus we get

$$\frac{1}{a}+\frac{1}{ab}+\frac{1}{abc}+\frac{1}{abcd}+\dots=\cfrac{1}{a-\cfrac{a}{b+1-\cfrac{b}{c+1-\cfrac{c}{d+1-\dots}}}}\ \dots\text{(III)}.$$

Putting $a=2$, $b=3$, $c=4$, $d=5$, ... we get

$$\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\ldots=\cfrac{1}{2-\cfrac{2}{4-\cfrac{3}{5-\cfrac{4}{6-\cfrac{5}{7-\ldots}}}}}$$

Hence $e=2+\cfrac{1}{2-\cfrac{2}{4-\cfrac{3}{5-\cfrac{4}{6-\cfrac{5}{7-\ldots}}}}}$

Also solved by G. W. Greenwood, and G. B. M. Zerr.

264. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Express the invariant $2(a_0a_4-4a_1a_3+3a_2^2)$ of the binary quartic $a_0x_1^4+4a_1x_1^3x_2+6a_2x_1^2x_2^2+4a_3x_1x_2^3+a_4x_2^4$ in terms of roots of the latter.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

It can be shown that, if

$$\begin{aligned}&a_0x_1^4+4a_1x_1^3x_2+6a_2x_1^2x_2^2-4a_3x_1x_2^3+a_4x_2^4\\&\equiv a_0(x_1^2+2px_1x_2+qx_2^2)(x_1^2+2p'x_1x_2+q'x_2^2),\end{aligned}$$

then $4\theta^3-I\theta+J=0$, where $I=a_0a_4-4a_1a_3+3a_2^2$, and $\theta=a_2-a_0pp'$.

Let β, γ be the roots of $x_1^2+2px_1x_2+qx_2^2=0$, and α, δ the roots of $x_1^2+2p'x_1x_2+q'x_2^2=0$. Then

$$\theta=\frac{a_0}{6}\Sigma\beta\gamma-\frac{a_0}{4}(\beta+\gamma)(\alpha+\delta)=\frac{a_0}{12}(v-w),$$

where $u=(\beta-\gamma)(\alpha-\delta)$, $v=(\gamma-\alpha)(\beta-\delta)$, $w=(\alpha-\beta)(\gamma-\delta)$. The roots of the reduced cubic are therefore,

$$\frac{a_0}{12}(u-v),\ \frac{a_0}{12}(v-w),\ \frac{a_0}{12}(w-u).$$

It is easily found that $u+v+w=0$. Consequently, $\Sigma vw=-\frac{1}{2}\Sigma u^2$, and

$$\frac{I}{4}=-\Sigma\theta_1\theta_2=-\frac{a_0^2}{144}\Sigma(uv-uw+vw-v^2)=\frac{a_0^2}{144}.\tfrac{3}{2}\Sigma u^2.$$

Hence $I=\dfrac{a_0^2}{24}\Sigma u^2$, where u, v, w have the values given above.

Also solved by G. B. M. Zerr.